MATH 245 F20, Final Exam
(120 minutes, open book, open notes)

1. Exam instructions.

2. (8pts) Consider the sequences $a_n = n^2+2^n$ and $b_n = 10n^3$. Select which of the following statements are true. (you may select as many as you wish, including none or all).
(i) $a_n = O(b_n)$; (ii) $a_n \neq O(b_n)$; (iii) $b_n = O(a_n)$; (iv) $b_n \neq O(a_n)$.

3. (8pts) Let S, T be sets. Select which of the following are sets. (you may select as many as you wish, including none or all).
(i) $S \cap T$; (ii) 2^S; (iii) $S \times T$; (iv) $|S|$; (v) $S \subseteq T$; (vi) a relation from S to T.

4. (8pts) Consider the relation $R = \{(1,2),(2,3),(3,3)\}$ on $S = \{1,2,3\}$. Select which of the following properties R satisfies. (you may select as many as you wish, including none or all).
(i) reflexive; (ii) irreflexive; (iii) symmetric; (iv) antisymmetric; (v) trichotomous; (vi) transitive.

5. (8pts) Consider the relation $R = \{(1,2),(2,3),(3,3)\}$ on $S = \{1,2,3\}$. Select which of the following properties R satisfies. (you may select as many as you wish, including none or all).
(i) left-total; (ii) right-total; (iii) left-definite; (iv) right-definite; (v) function; (vi) bijection.

6. (12pts) Prove or disprove: For all propositions p, q, we have $p \oplus q \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$.

7. (12pts) Prove or disprove the following statement: $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \lfloor x \rfloor = \lfloor 2y \rfloor \to x = 2y$.

8. (12pts) Prove that, for every natural number n, we have $\binom{3n}{n} \leq 7^n$.

9. (12pts) Let S, T be sets. Carefully state the converse of: If $S \subseteq T$, then $S \setminus T \subseteq T$. Then, prove or disprove your statement.

10. (12pts) Prove or disprove that, for all nonempty sets S, T, we must have $|S| \leq |S \times T|$.

11. (12pts) Define relation R on $\mathbb{N}$ via $R = \{(a,b) : \frac{a}{b} \in \mathbb{Z}\}$. Prove or disprove that R is an equivalence relation.

12. (12pts) Find all solutions $x \in [0, 128)$ satisfying $12x \equiv 20 \pmod{128}$. Justify your calculations.

13. (12pts) Prove or disprove: For all $x, y \in \mathbb{Z}$, if $x \equiv y \pmod{240}$, then $x \equiv y \pmod{18}$.

14. (12pts) Consider the equivalence relation $\equiv_7$ on $S = \{1, 2, 3, \ldots, 100\}$. Determine $|[3]| + |[2]|$.

15. (12pts) Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}$. S is the set of 2×2 matrices with integer coefficients. Define relation R on S via $R = \left\{ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) : a \leq a' \wedge b \leq b' \wedge c \leq c' \wedge d \leq d' \right\}$.
Prove that R is a partial order.
Note: You do not need to know anything about matrices to solve this problem, except that they are some numbers arranged in a grid with square brackets to the left and right.

16. (12pts) Just as in the previous question, let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}$, and define relation R on S via $R = \left\{ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) : a \leq a' \wedge b \leq b' \wedge c \leq c' \wedge d \leq d' \right\}$.
Draw the Hasse diagram for the interval poset $\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right]$. You may assume that R is a partial order (as proved in the previous question).

17. (12pts) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find a partial order on S that has width 5 and height 5. Give your answer in the form of a Hasse diagram, and justify its width and height.

18. (12pts) Let $S = \{1, 2, 3, 4, 5, 6\}$. Find a relation on S that is left-total, right total, not left-definite, and not right-definite. Give your answer as a set of ordered pairs, and justify the four properties listed.

19. (12pts) Consider the function $R = \{(x, y) : y = \frac{2x}{x^2+1}\}$ on $\mathbb{R}$. Prove or disprove that R is injective.